Final Exam Mathematical Physics, Prof. G. Palasantzas

- Total number of points 100
- 10 points for coming to the final exam
- Justify your answers for all problems

Problem 1 (10 points)

Show that $\lim\limits_{n\to\infty} \dfrac{x^n}{n!} = 0$ for all x $\in$ (-∞, +∞)

Problem 2 (15 points)

Consider the series $\sum_{n=1}^{+\infty} \frac{(-5)^n}{n^{5/2}} x^n$

For which values of x does the series converge?

Problem 3 (15 points)

Suppose a mass m is attached to a spring with spring constant k ($=m\omega^2$). If an external force $F(t)=F_o \cos(\omega t)$ is applied to the mass m, then its equation of motion is given by:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t)$$

Suppose $c^2 - 4mk < 0$, then prove that the motion of the mass m is described by the general solution*:

$$x(t) = e^{-(c/2m)t}\left[c_1\cos(\tilde{\omega}t) + c_2\sin(\tilde{\omega}t)\right] + \left(\frac{F_o}{c\omega}\right)\sin(\omega t)$$

with $\tilde{\omega} = \omega\sqrt{1-(c/2m\omega)^2}$ and c_1, c_2 constants.

**You have to derive the general solution using the method of undetermined coefficients.*

Problem 4 (15 points)

Find the periodic solution in complex form of the second order differential equation:

$$M\frac{d^2X}{dt^2} + C\frac{dX}{dt} + KX = F(t)$$

F(t) a known periodic function with period T. M, C and K are real positive constants.

To this end consider the complex Fourier series definitions $X(t) = \sum_{n=-\infty}^{+\infty} X_n e^{in\omega t}$, and $F(t) = \sum_{n=-\infty}^{+\infty} F_n e^{in\omega t}$ with $\omega=2\pi / T$

Problem 5 (15 points)

Consider the Fourier expression for the Dirac delta function: $\delta(x) = \int_{-\infty}^{+\infty} e^{-i2\pi kx} dk$

(a: 5 points) Prove that $\delta(ax) = \delta(x)/|a|$ ($a\neq0$)

(b: 10 points) Calculate the double integral $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-x^2} e^{-i2\pi kx} \cos(6\pi k)\, dxdk$

Problem 6 (20 points)

Assume a function f(x) to have the Fourier transform: $F(k) = \int_{-\infty}^{+\infty} f(x)\, e^{-i2\pi kx} dx$

Consider the Fourier expression of the Dirac Delta function: $\delta(k) = \int_{-\infty}^{+\infty} e^{-i2\pi kx} dx$

(a: 10 points) Derive the Fourier Transform of $f(x) = \cos(9\pi k_o x)$

(b: 10 points) Derive the Fourier Transform of $f(x) = \sin^2(9\pi k_o x)$

Problem 1

We form the series $\sum_{n=1}^{\infty} \frac{x^n}{n!}$, and we show that it is covergent using the ratio test.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \frac{|x|}{n+1} \to 0 < 1 \quad \text{with } x \in (-\infty, +\infty)$$

Therefore, since the series is convergent we have: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{x^n}{n!} = 0$

Problem 2

If $a_n = (-5)^n x^n / n^{5/2}$, take the ratio test

~~then~~

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-5)^{n+1}x^{n+1}}{(n+1)^{5/2}}}{\frac{(-5)^n x^n}{n^{5/2}}}\right| = 5|x| \lim_{n\to\infty}\frac{1}{(1+\frac{1}{n})^{5/2}} = 5|x|$$

thus we have $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 5|x|$

- By the ratio test it converges if $5|x|<1$
  $\Rightarrow \quad |x| < \frac{1}{5}$
- For $x = \frac{1}{5}$ we have the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{5/2}}$ converges as alternating series or absolutely convergent p-series with $p=5/2>1$ so it is convergent
- For $x=-1/5$ we have $\sum_{n=1}^{\infty}\frac{1}{n^{5/2}}$ convergent p-series with $p=5/2>1$

Thus the interval of convergence is

$$\left[-1/5,\ 1/5\right]$$

Problem 3

CASE III $c^2 - 4mk < 0$ **(underdamping)**
Here the roots are complex:

If we solve the auxiliary equation we have two complex roots:

$$r_{1,2} = -(c/2m) \pm j\omega\sqrt{1-(c/2m\omega)^2} \quad (k = m\omega^2) \Rightarrow$$

$$\text{Homogenous solution}: x_{\text{homo}}(t) = e^{-(c/2m)t}\left[c_1\cos(\tilde{\omega}t) + c_2\sin(\tilde{\omega}t)\right]$$

$$\text{with } \tilde{\omega} = \omega\sqrt{1-(c/2m\omega)^2}$$

(see book chap. 17.3 & Nestor site)

We look for a particular solution of the form: $x_p(t) = \mathrm{A}\cos(\omega t) + \mathrm{B}\sin(\omega t)$

Look in Nestor the general solution of the AFM-equation of motion with $\omega=\omega_o$:

since $\omega = \omega_o$ (and as a result $k - m\omega^2 = 0$) we obtain after substition into the equation of motion : $c\omega\mathrm{B} = \mathrm{F}_o$ and $\mathrm{A} = 0$

$$\Rightarrow x_p(t) = \left(\frac{F_o}{c\omega}\right)\sin(\omega t)$$

Total solution : $x(t) = x_{\text{homo}}(t) + x_p(t)$

$$x(t) = e^{-(c/2m)t}\left[c_1\cos(\tilde{\omega}t) + c_2\sin(\tilde{\omega}t)\right] + \left(\frac{F_o}{c\omega}\right)\sin(\omega t)$$

Problem 4

We take the derivatives of $X(t)$

$$X'(t) = \sum_{n=-\infty}^{+\infty} X_n (in\omega)\, e^{in\omega t}$$

$$X''(t) = \sum_{-\infty}^{+\infty} X_n [-(n\omega)^2]\, e^{in\omega t}$$

Substitute into the differential equation

$$M \sum_{-\infty}^{+\infty} X_n [-(n\omega)^2]\, e^{in\omega t} + C \sum_{-\infty}^{+\infty} X_n (in\omega) e^{in\omega t}$$

$$+ K \sum_{-\infty}^{+\infty} X_n \, e^{in\omega t} = \sum_{-\infty}^{+\infty} f_n\, e^{in\omega t} \Rightarrow$$

$$\sum_{n=-\infty}^{+\infty} \left[X_n \left\{ [K - (n\omega)^2 M] + i n\omega C \right\} - f_n \right] e^{in\omega t} = 0$$

$$\Rightarrow \quad X_n \left\{ [K - M(n\omega)^2] + i n\omega C \right\} - f_n = 0$$

$$\Rightarrow \quad X_n = \frac{f_n}{[K - M(n\omega)^2] + i n\omega C}$$

Thus the solution has the form

$$X(t) = \sum_{n=-\infty}^{+\infty} \frac{f_n}{[K - M(n\omega)^2]\ i n\omega C}\, e^{in\omega t}$$

$$f_n = \frac{1}{T} \int_{-T/2}^{T/2} F(t)\, e^{-in\omega t}\, dt$$

Problem 5

(a)

$$\delta(\alpha k) = \int_{-\infty}^{+\infty} e^{-i2\pi k\alpha x}\,dx$$

I) $\alpha > 0$ change variables $y = \alpha x$

$$\delta(\alpha k) = \frac{1}{\alpha}\int_{-\infty}^{+\infty} e^{-i2\pi k y}\,dy = \frac{1}{\alpha}\delta(k)$$

II) $\alpha < 0$ change variables $y = \alpha x$

$$\delta(\alpha k) = \frac{1}{\alpha}\int_{+\infty}^{-\infty} e^{-i2\pi k y}\,dy = -\frac{1}{\alpha}\delta(k)$$

Thus we have $\delta(\alpha k) = \delta(k)/|\alpha|$

(b)

$$\int_{-\infty}^{+\infty} \mathrm{dx} \int_{-\infty}^{+\infty} \mathrm{dk}\, \mathrm{e}^{-\mathrm{x}^2} \mathrm{e}^{-\mathrm{i}2\pi \mathrm{kx}} \cos(6\pi \mathrm{k})$$

$$= \int_{-\infty}^{+\infty} \mathrm{dx}\, \mathrm{e}^{-\mathrm{x}^2} \int_{-\infty}^{+\infty} \mathrm{dk}\, \mathrm{e}^{-\mathrm{i}2\pi \mathrm{k(x)}} (\mathrm{e}^{\mathrm{i}2\pi \mathrm{k}(3)} + \mathrm{e}^{-\mathrm{i}2\pi \mathrm{k}(3)})/2 =$$

$$\int_{-\infty}^{+\infty} \mathrm{dx}\, \mathrm{e}^{-\mathrm{x}^2} \int_{-\infty}^{+\infty} \mathrm{dk}\, (\mathrm{e}^{-\mathrm{i}2\pi \mathrm{k}(\mathrm{x}-3)} + \mathrm{e}^{-\mathrm{i}2\pi \mathrm{k}(\mathrm{x}+3)})/2$$

From the definition of the delta function we have $\int_{-\infty}^{+\infty} \mathrm{dk}\, \mathrm{e}^{-\mathrm{i}2\pi \mathrm{k}(\mathrm{x}\pm 3)} = \delta(x \pm 3)$

$$\int_{-\infty}^{+\infty} \mathrm{dx} \int_{-\infty}^{+\infty} \mathrm{dk}\, \mathrm{e}^{-\mathrm{x}^2} \mathrm{e}^{-\mathrm{i}2\pi \mathrm{kx}} \cos(6\pi \mathrm{k})$$

$$= (1/2)\{\int_{-\infty}^{+\infty} \mathrm{dx}\, \mathrm{e}^{-\mathrm{x}^2}\, \delta(x-3) + \int_{-\infty}^{+\infty} \mathrm{dx}\, \mathrm{e}^{-\mathrm{x}^2}\, \delta(x+3)\} = e^{-9}$$

Probblem 6

(a)

$$F(k) = \int_{-\infty}^{+\infty} \cos(9\pi k_0 x)\, e^{-i 2\pi k x} dx =$$

$$= \frac{1}{2} \int_{-\infty}^{+\infty} \left[e^{i(9\pi k_0 x)} + e^{-i(9\pi k_0 x)} \right] e^{-i 2\pi k x} dx =$$

$$= \frac{1}{2} \left\{ \underbrace{\int_{-\infty}^{+\infty} e^{-i 2\pi \left(k - \frac{9k_0}{2}\right)x} dx}_{\delta\left(k - \frac{9k_0}{2}\right)} + \underbrace{\int_{-\infty}^{+\infty} e^{i 2\pi \left(k + \frac{9k_0}{2}\right)x} dx}_{\delta\left(k + \frac{9k_0}{2}\right)} \right\}$$

$$\Rightarrow F(k) = \frac{1}{2}\left[\delta\left(k - \frac{9k_0}{2}\right) + \delta\left(k + \frac{9k_0}{2}\right) \right]$$

(b)

$$\sin^2(9\pi k_0 x) = \{1 - \cos(18\pi k_0 x)\}/2$$

$$F(k) = \int_{-\infty}^{+\infty} \sin^2(9\pi k_0 x)\, e^{-i 2\pi k x} dx =$$

$$= \frac{1}{2} \underbrace{\int_{-\infty}^{+\infty} e^{-i 2\pi k x} dx}_{\delta(k)} - \underbrace{\frac{1}{2} \int_{-\infty}^{+\infty} \cos(18\pi k_0 x)\, e^{-i 2\pi k x} dx}$$

From (b) if we replace k_0 with $2k_0$ we have $\frac{1}{2}\left[\delta(k - 9k_0) + \delta(k + 9k_0)\right]$

Thus we have

$$F(k) = \frac{1}{2}\delta(k) - \frac{1}{4}\left[\delta(k - 9k_0) + \delta(k + 9k_0)\right]$$